1. Banach Space

A Banach space is a complete normed space. The definition of a complete normed space is as follows:

(1) Vector Spaces

A *vector space*(or a *linear space*) over is a set of objects(called *vectors*) which can be added together or multiplied by scalars in

More formally, a vector space V over set V with two operations: *vector addition* and *scalar multiplication*. With these operations given, the vector space V must satisfy the following axioms:

For each **u,v,w** V and

A1 (Associativity of addition): **u** + (**v** + **w**) = (**u** + **v**) + **w**

A2 (Commutativity of addition): **u** + **v** = **v** + **u**

A3 (Identity element of addition): There exists an element **0**  V, called the zero vector, such that **u** + **0** = **u** for all **u** V

A4 (Inverse elements of addition): For every **u** V, there exists an element −**u** V, called the additive inverse of **u**, such that **u** + (−**u**) = **0**

A5 (Compatibility of scalar multiplication with field multiplication):

A6 (Identity element of scalar multiplication): 1**u** = **u**, where 1 denotes the multiplicative identity in

A7 (Distributivity of scalar multiplication with respect to vector addition):

A8 (Distributivity of scalar multiplication with respect to field addition):

(2) Norms on vector spaces

Let V be a vector space over . A *norm* on the vector space V is a function satisfying the following.

A vector space with norm is called a *normed vector space*.

(3) Convergent sequences, Cauchy sequences

Let

Then,

In this case, we denote

(4) Banach Space

A vector space V over is called to be *complete* if every Cauchy sequence is convergent.

A *complete normed vector space* is also called a *Banach space*.

2. Closed sets

Given is called an *open ball* centered at **x** with radius .

A subset

A subset M of V is called to be *closed* if is open.

Each closed subset M of V satisfies the following property:

If

3. Operators

Let M and Y be sets. An *operator* associates to each point u in M to a point v in Y denoted by v=Au.

The set M is called the *domain* of A, we also write M=D(A).

The set A(M) is called the *range* of A.

Operators are also called functions.

4. The Banach Fixed Point Theorem

We assume that:

Then the following hold true:

(i) **Existence and uniqueness.** The equation

(ii) **Convergence of the iteration method.**

Proof)

Step 1: We show first that

Let n=1,2,… . Since

Now let n=0,1,… and m=1,2,… The triangular inequality and the sum formula for the geometric series yield

Since X is a Banach space, the Cauchy sequence converges, i.e.,

Step 2: We show that the limit point u is a solution of the equation

From

Since the set M is closed, we obtain

,and hence . By

Hence , proving (2).

Step 3: We show the uniqueness of the solution u of . Suppose u and v are solutions. Then

, proving (1).